The Effectiveness of PIES Compared to BEM in the Modelling of 3D Polygonal Problems Defined by Navier-Lame Equations

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***Abstract*—The paper presents the effectiveness of the parametric integral equation system (PIES) in solving 3D boundary problems defined by Navier-Lame equations on the example of the polygonal boundary and in comparison with the boundary element method (BEM). Analysis was performed using the commercial software "BEASY" which bases on the BEM method, whilst in the case of PIES using an authors software. On the basis of two examples the number of data necessary to modeling of the boundary geometry, the number of solved algebraic equations and the stability and accuracy of the obtained numerical results were compared.**

***Index Terms*—computer modeling, computer simulation, boundary problems, parametric integral equation system (PIES), Navier-Lame equation**

I. Introduction

The most popular in the computer simulation of three-dimensional boundary problems are the finite (FEM) [1,13] and boundary (BEM) [1,2,3] element methods. Their popularity comes from the variety of applications, but also from the wide availability of tested and therefore reliable software [4]. Both methods are characterized by the necessity of discretization at the stage of modeling the shape: in FEM of the boundary and the area by finite elements, in BEM only of the boundary using boundary elements. On the one hand this is advantageous, because using any number of such elements varied and complex shapes can be modeled, however, particularly in the case of 3D issues it is very time consuming and complicated. Another disadvantage is that obtained by these methods solutions are discrete, except that every time we get the number of solutions from boundary nodes (BEM, FEM) or from area (FEM), which are useless to us. Our previous research on solving boundary problems resulted in the creation and development of the parametric integral equation system (PIES) [7,8,9,10,11,12], which eliminates disadvantages mentioned above. It is an analytical modification of the boundary integral equation (BIE), which is solved using BEM. BEM has already substantially reduced the complexity of the boundary problems solving, because of the limitation of the discretization only to the boundary, so the next step is to completely get rid of the discretization. This has been achieved by analytical including the shape of the boundary in the mathematical formalism of BIE [7]. That shape, thanks to the separation of the boundary geometry approximation from the approximation of boundary functions, can be defined in any manner. The most effective is the use of parametric curves in 2D problems and parametric surfaces in the case of three-dimensional problems [5,6]. The application of surfaces (e.g. Coons or Bézier) for 3D issues eliminates the need for the discretization of individual faces of the three-dimensional geometry, because they can be globally modeled with a single surface. The accuracy of PIES solutions no longer depends on the method of modeling the shape (if we assume that we model the exact shape using surface patches), only on the effectiveness of the method used for the approximation of boundary functions. Solutions in PIES are presented in the form of approximation series with any number of coefficients, which however means that they are available in the continuous version. PIES has been widely tested, taking into account the various problems modeled by 2D differential equations (Laplace's, Navier-Lame, Helmholtz) [7,10,9], but also modeled using three-dimensional Laplace [8] and Helmholtz [11] equations. The results of these tests were satisfactory, and therefore the study were extended on more complex issues, namely the problems of elasticity. Preliminary analysis of such problems on the example of polygonal issues was presented in [12]. The purpose of this study is to examine the effectiveness of the numerical implementation of PIES for solving Navier-Lame equations in the 3D area compared with the classical BEM. The authors also tried to answer the question: can PIES be a potential alternative to classical element method. For the implementation of BEM the commercial software "BEASY" was used, whilst in the case of PIES the authors software. Compared were: the number of input data needed to define the shape of the boundary, the number of solved algebraic equations and the accuracy, stability and reliability of results. Presented tests are beginning of research in this direction and are limited to a very elementary form of the area.

II. PIES in 3D Elasticity Problems

PIES is an analytic modification of the classical boundary integral equations (BIE), which are characterized by the fact that the boundary is defined generally by using the boundary integral. BEM used to solve BIE is characterized by the discretization of the boundary, as physical and effective way of its definition. Such definition of the boundary has also some disadvantages, because any minor change requires a re-discretization. Analytical modification of BIE consisted in defining the boundary not using the boundary integral, but

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with surfaces used in computer graphics. These patches have been integrated into kernels of obtained PIES. PIES therefore is not defined on the boundary just as BIE, but on the parametric reference surface. Such a definition of PIES has the advantage that changing the shape of the boundary causes automatic adjustment of PIES to the modified shape and does not require classical discretization. The way of obtaining PIES for the Navier-Lame equations in 3D areas is a generalization of the technique discussed in details in [10] for 2D problems. PIES for considered in the paper problems is presented in the following form [12]

$$0.5\boldsymbol{u}_l(v_1,w_1)=\sum_{j=1}^{n}\int_{v_{j-1}}^{v_j}\int_{w_{j-1}}^{w_j}\{\overline{\boldsymbol{U}}_{lj}^{*}(v_1,w_1,v,w)\boldsymbol{p}_j(v,w)-\overline{\boldsymbol{P}}_{lj}^{*}(v_1,w_1,v,w)\boldsymbol{u}_j(v,w)\}J_j(v,w)dvdw, \quad (1)$$

where $v_{l-1}<v_1<v_l$, $w_{l-1}<w_1<w_l$, $v_{j-1}<v<v_j$, $w_{j-1}<w<w_j$, $l=1,2,3...,n$ and n is the number of surfaces which define 3D geometry.

Integrands $\overline{\boldsymbol{U}}_{lj}^{*}(v_1,w_1,v,w)$, $\overline{\boldsymbol{P}}_{lj}^{*}(v_1,w_1,v,w)$ from (1) are presented in the following matrix form

$$\overline{\boldsymbol{U}}_{lj}^{*}(v_1,w_1,v,w)=\frac{1}{16\pi(1-\nu)\mu\eta}\begin{bmatrix}U_{11} & U_{12} & U_{13}\\ U_{21} & U_{22} & U_{23}\\ U_{31} & U_{32} & U_{32}\end{bmatrix}, \quad (2)$$

$$\overline{\boldsymbol{P}}_{lj}^{*}(v_1,w_1,v,w)=\frac{-1}{8\pi(1-\nu)\eta^2}\begin{bmatrix}P_{11} & P_{12} & P_{13}\\ P_{21} & P_{22} & P_{23}\\ P_{31} & P_{32} & P_{33}\end{bmatrix}. \quad (3)$$

The individual elements in the matrix (2) in the explicit form can be presented

$$U_{11}=(3-4\nu)+\frac{\eta_1^2}{\eta^2},\ U_{12}=\frac{\eta_1\eta_2}{\eta^2},\ U_{13}=\frac{\eta_1\eta_3}{\eta^2},$$

$$U_{21}=\frac{\eta_2\eta_1}{\eta^2},\ U_{22}=(3-4\nu)+\frac{\eta_2^2}{\eta^2},\ U_{23}=\frac{\eta_2\eta_3}{\eta^2},$$

$$U_{31}=\frac{\eta_3\eta_1}{\eta^2},\ U_{32}=\frac{\eta_3\eta_2}{\eta^2},\ U_{33}=(3-4\nu)+\frac{\eta_3^2}{\eta^2},$$

whilst in the matrix (3) are presented by

$$P_{11}=((1-2\nu)+3\frac{\eta_1^2}{\eta^2})\frac{\partial\eta}{\partial n},\ P_{22}=((1-2\nu)+3\frac{\eta_2^2}{\eta^2})\frac{\partial\eta}{\partial n},$$

$$P_{33}=((1-2\nu)+3\frac{\eta_3^2}{\eta^2})\frac{\partial\eta}{\partial n},$$

$$P_{12}=3\frac{\eta_1\eta_2}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_1n_2-\eta_2n_1}{\eta},$$

$$P_{13}=3\frac{\eta_1\eta_3}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_1n_3-\eta_3n_1}{\eta},$$

$$P_{21}=3\frac{\eta_2\eta_1}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_2n_1-\eta_1n_2}{\eta},$$

$$P_{23}=3\frac{\eta_2\eta_3}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_2n_3-\eta_3n_2}{\eta},$$

$$P_{31}=3\frac{\eta_3\eta_1}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_3n_1-\eta_1n_3}{\eta},$$

$$P_{32}=3\frac{\eta_3\eta_2}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_3n_2-\eta_2n_3}{\eta},$$

where function $J_j(v,w)$ is the Jacobian, n_1,n_2,n_2 are components of the normal vector $\boldsymbol{n}_j$ to the segment of the boundary marked by j, whilst

$\eta_1=P_j^{(1)}(v,w)-P_l^{(1)}(v_1,w_1)$, $\eta_2=P_j^{(2)}(v,w)-P_l^{(2)}(v_1,w_1)$,

$\eta_3=P_j^{(3)}(v,w)-P_l^{(3)}(v_1,w_1)$ and $\eta=\left[\eta_1^2+\eta_2^2+\eta_3^2\right]^{0.5}$,

where $P_j^{(1)}(v,w)$, $P_j^{(2)}(v,w)$, $P_j^{(3)}(v,w)$ are the scalar components of the vector surface $\boldsymbol{P}_j(v,w)=[P_j^{(1)}(v,w),P_j^{(2)}(v,w),P_j^{(3)}(v,w)]^T$ which depends on (v,w) parameters. This clause is also valid for the surface marked by l with parameters v_1w_1, i.e. for $j=l$ and parameters $v=v_1$ and $w=w_1$.

III. Numerical modeling of 3D elasticity problems by PIES

In view of the separation in PIES the approximation of the boundary geometry from boundary functions we deal with perfecting both these approximations independently. Thus, we use the way of the modeling of the boundary geometry - taken from computer graphics - which consists in modeling the boundary by surfaces. Such modeling requires the least number of the necessary data.

A. Modeling of the three-dimensional boundary geometry

The proposed in the paper method is characterized by the fact that in the mathematical formalism of PIES the boundary geometry is directly included, and it can be physically defined by a suitable connection (arbitrary) of parametric surfaces. This strategy and its effectiveness has been tested considering issues defined by the Laplace and Hemholtza equations [8,11]. Both polygonal and curvilinear shapes were tested, to define which were used respectively Coons and Bézier surfaces. This paper, in connection with the initial stage of research in solving 3D problems defined by Navier-Lame equations, is limited to examining polygonal areas, hence the description of modeling the boundary geometry will concern only such cases. Modeling of polygonal areas in PIES is performed in very simple, intuitive and effective way. Each face of the 3D area is defined by the Coons surface, which require posing only four its corner points. It is a minimal set

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of data necessary to the accurate definition of considered shape. The strategy of the global modeling in PIES on the example of the cube with a comparison to BEM is presented in Fig.1.

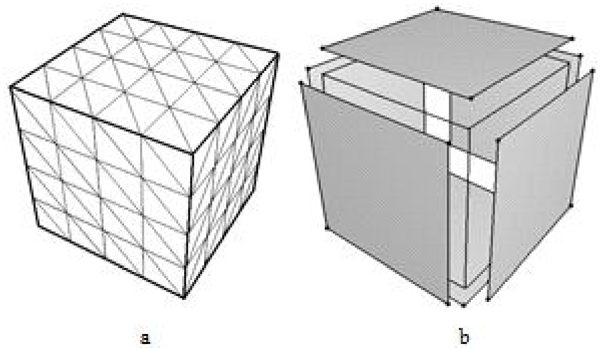

Figure 1. Modeling of the 3D boundary geometry: a) in BEM, b) in PIES

Presented in Fig.1 cube was modeled: in BEM (Fig.1a) by discretization of its faces using together 192 triangular boundary elements, whilst in PIES (Fig.1b) using only 8 corner points which define 6 Coons surfaces which model cube faces. On the basis of presented figure the efficiency of the proposed modeling technique is clearly seen, and manifests itself in reduced workload related to the modeling of the boundary geometry and eliminating the need for discretization.

B. *Approximation of boundary functions*

Besides the modeling of the boundary geometry in PIES the second problem is its solving. As it was repeatedly emphasized in other papers [7,8,9,10,11], the solution is practically reduced to the approximation of boundary functions on individual surfaces. For that reason we use approximation series with arbitrary base functions (for 2D problems Chebyshev, Laguerre, Legendre and Hermite polynomials were tested). For the each surface which model j segment of the boundary they are presented in the following form [11]

$$\begin{aligned} \boldsymbol{u}_j(v,w) &= \sum_{p=0}^{N}\sum_{r=0}^{M} \boldsymbol{u}_j^{(pr)} T_j^{(p)}(v) T_j^{(r)}(w), \\ \boldsymbol{p}_j(v,w) &= \sum_{p=0}^{N}\sum_{r=0}^{M} \boldsymbol{p}_j^{(pr)} T_j^{(p)}(v) T_j^{(r)}(w), \end{aligned} \qquad (4a,b)$$

where $\boldsymbol{u}_j^{(pr)}, \boldsymbol{p}_j^{(pr)}$ are unknown coefficients, $T_j^{(p)}(v)$, $T_j^{(r)}(w)$ are Chebyshev polynomials of the first kind, whilst N, M - are numbers of these coefficients on the surface in two directions (together $k = N \times M$).

Such technique is much more effective then re-discretization in BEM, because does not require modification of the defined geometry and is reduced to manipulation of two parameters N and M in the computer software. After solving PIES only solutions at the boundary are obtained. To receive the results also in the area the integral identity should be used (Appendix A). The method of receipt is the same as in the case of 2D problems, presented in [10]. In Appendix B, are also included formulas used to calculate stresses in the authors software with integrands in explicit form.

IV. Analysis of Results

In the first example we consider the unit cube presented in Fig.1. One face of the cube is firmly fixed and the opposite is subjected to the surface compressive force $p_z = 1MPa$. Material constants adopted for the calculation are: the Young's modulus $E = 1MPa$ and the Poisson's ratio $\nu = 0.3$. In order to define the shape of the boundary in PIES we use rectangular Coons surfaces. Their definition is very effective and intuitive and is limited to posing the coordinates of the corner points of each surface that creates the boundary of the cube. As shown in Fig. 1b, modeling of the boundary geometry leads to the use of six Coons surfaces and posing 8 corner points of these surfaces. In the case of BEM action would be much more laborious, because would require the discretization of the boundary by boundary elements, whose number would depend on the desired accuracy of the solutions. PIES is characterized by the fact of the separation of the approximation of the boundary geometry from boundary functions and, therefore, modeling is carried out in a very effective way, because with minimal number of data required to the accurate mapping of the shape. Thus, once defined geometry is constant throughout the process of numerical calculations. The accuracy of the solutions in PIES depends on the number of expressions in series which approximate boundary functions (4a, b). It is easy to change parameter k, and the values used for the solution of the analyzed example, are as follows (for each surface separately): 16, 25, 36, 49, 64, 81 and 100. The greater value the more algebraic equations we have finally to be solved, but also the more accurate solutions. Thus, when we choose value 100, to obtain the final results we must solve the system with 1800 algebraic equations. The same example is also solved using the "BEASY" [4], representing the BEM. In order to analyze the accuracy and stability of solutions the boundary geometry was modeled using a different number and various kinds of boundary elements. Therefore, we have used constant, linear and quadratic boundary elements in the total number of 24, 54, 96, 150, 294, 600 and 864. The stabilization of the results (no change to third decimal place) was obtained using linear and square elements in the number of 600 and more. Assuming that we take into account 600 square elements, it is necessary to solve 6138 algebraic equations. It is over 3 times more than in the method developed by the authors. Table I shows the number of data needed to define the shape from Fig. 1 using two mentioned methods by which the obtained solutions are stable. The comparison of the number of data is more beneficial for PIES. It comes from the separation of the approximation of the boundary geometry from boundary functions and from the fact that we use the efficient way of the modeling of the boundary and the effective method for the approximation of boundary functions.

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TABLE I.
COMPARISON OF THE EFFECTIVENESS OF MODELING IN BEM AND PIES

	BEM	PIES
Number and kind of data for the boundary geometry definition	600 quadratic boundary elements and 5400 nodes	8 corner points of surfaces
Number of solved algebraic equations	6138	1800
Obtained results	in 3069 nodes on the boundary and arbitrary points in the domain	in arbitrary points on the boundary and in the domain

Despite the benefits of PIES in the reduced number of data for modeling and the number of algebraic equations in the system of equations we must consider especially the accuracy of solutions. For this reason we analyze displacements u_z and stresses σ_x, σ_z in two cross-sections of the considered area: $x = 0,\ \ y = 0, -0.5 < z < 0.5$ and $x = 0,\ \ -0.5 < y < 0.5, z = 0.3$. The results obtained by comparison of PIES solutions with solutions achieved using BEM (“BEASY”) are presented in Table II and Fig. 2.

TABLE II.
COMPARISON OF DISPLACEMENTS u_z OBTAINED BY BEM AND PIES

x	y	z	BEM	PIES
0.0	0.0	-0.4	-0.07167	-0.072275
0.0	0.0	-0.3	-0.15635	-0.157760
0.0	0.0	-0.2	-0.25082	-0.253072
0.0	0.0	-0.1	-0.35114	-0.354127
0.0	0.0	0.0	-0.45406	-0.457618
0.0	0.0	0.1	-0.55754	-0.561490
0.0	0.0	0.2	-0.66053	-0.664752
0.0	0.0	0.3	-0.76271	-0.767112
0.0	0.0	0.4	-0.86420	-0.868726
0.0	-0.4	0.3	-0.76976	-0.774406
0.0	-0.3	0.3	-0.76755	-0.772130
0.0	-0.2	0.3	-0.76515	-0.769648
0.0	-0.1	0.3	-0.76336	-0.767793
0.0	0.1	0.3	-0.76336	-0.767793
0.0	0.2	0.3	-0.76515	-0.769648
0.0	0.3	0.3	-0.76755	-0.772130
0.0	0.4	0.3	-0.76976	-0.774406

As shown in Table II and Fig. 2 obtained by the proposed method results are very similar to the solutions obtained using BEM. It should be emphasized, however, that the results obtained by the authors require less workload related to the modeling (a small set of corner points) and numerical solution (over 3 times less number of algebraic equations in the equation system). Fig. 3 shows how the values of displacements u_z in the chosen point $(0,0,-0.4)$ become stable, taking into account both methods and different values of parameters responsible for improving the accuracy of the solutions (eventually expressed as a number of algebraic equations).

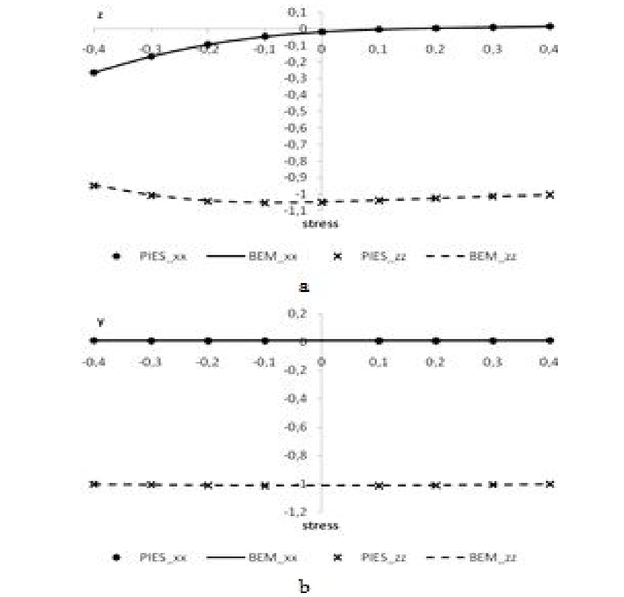

Figure 2. Comparison of normal stresses in cross-sections: a) $x = 0, y = 0, -0.5 < z < 0.5$ and b) $x = 0, -0.5 < y < 0.5, z = 0.3$.

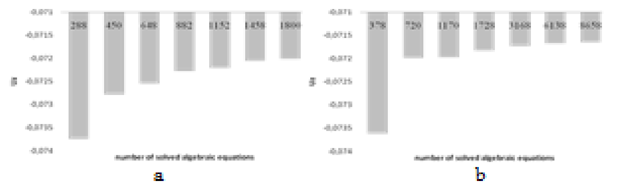

Figure 3. Displacements for different size of the algebraic equation system in: a) PIES and b) BEM

As shown in Fig.3, the stabilization of solutions with the desired accuracy in PIES took place at a much smaller number of data necessary to solve the problem than in BEM. As mentioned earlier it requires more than 3 times smaller system of algebraic equations then in the classical element method. In order to confirm the reliability of the proposed method and its effectiveness we solve the similar problem by changing only the boundary conditions. Thus, for the problem we assume the same shape and dimensions, the same material constants and the same degree of discretization in MEB and the number of expressions in approximation series in PIES. These assumptions caused that the data from Table 1 are valid also for the second considered problem. This time, however, the cube is subjected to the load in the form of the surface force $p_y = 1MPa$.

The values of solutions obtained by BEM and PIES were analyzed again. One previously mentioned cross-section was chosen ($x = 0, y = 0, -0.5 < z < 0.5$), and research was related to functions such as: displacements u_y and stresses σ_{yz}. The results are presented in Fig. 4.

As in the previous example, obtained by PIES solutions are practically the same as those obtained using BEM. So we can consider them to be reliable, due to the fact that BEM is the method developed through years and widely tested. Simultaneously, the same level of accuracy of solutions was obtained with: a smaller number of data for modeling, the lack

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of need for discretization the boundary, a smaller number of equations in the final system of algebraic equations and possibility for obtaining solutions at any point on the boundary and in the domain, without necessity of maintaining the results from all boundary nodes.

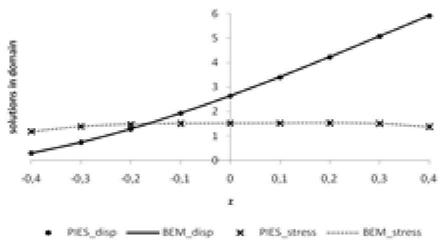

Figure 4. Displacements and stresses obtained by BEM and PIES in the chosen cross-section

Conclusions

The paper presents the efficiency of PIES in solving three-dimensional polygonal boundary problems defined by Navier-Lame equations in comparison to the classic BEM represented by “BEASY” software. Obtained by PIES solutions (both displacements and stresses) are characterized by high accuracy in comparison with the element method, but also are obtained in a much more efficient way. It was shown that the proposed approach requires fewer data to model the boundary geometry, the system with fewer number of equations and unknowns is solved and does not require to maintain redundant solutions collected from all grid nodes, but it is possible to obtain them at any point of the boundary or area. The authors are aware of the fact that the analyzed example is very basic, but getting even these results require intensive workload. The reliability of obtained results is encouraging to generalize the method to more complex shapes of areas.

Appendix A Displacements in a domain

A modified integral identity for displacements takes the following form

$$\boldsymbol{u}(\boldsymbol{x})=\sum_{j=1}^{n}\int_{v_{j-1}}^{v_j}\int_{w_{j-1}}^{w_j}\{\hat{\bar{\boldsymbol{U}}}_j^{*}(\boldsymbol{x},v,w)\,\boldsymbol{p}_j(v,w)-\hat{\bar{\boldsymbol{P}}}_j^{*}(\boldsymbol{x},v,w)\boldsymbol{u}_j(v,w)\}J_j(v,w)\,dv\,dw. \tag{5}$$

Integrands from (5) are presented by

$$\hat{\bar{\boldsymbol{U}}}_j^{*}(\boldsymbol{x},v,w)=\frac{1}{16\pi(1-\nu)\mu\vec{r}}\begin{bmatrix}\hat{U}_{11} & \hat{U}_{12} & \hat{U}_{13}\\ \hat{U}_{21} & \hat{U}_{22} & \hat{U}_{23}\\ \hat{U}_{31} & \hat{U}_{32} & \hat{U}_{33}\end{bmatrix}, \tag{6}$$

$$\hat{\bar{\boldsymbol{P}}}_j^{*}(\boldsymbol{x},v,w)=\frac{-1}{8\pi(1-\nu)\vec{r}^{\,2}}\begin{bmatrix}\hat{P}_{11} & \hat{P}_{12} & \hat{P}_{13}\\ \hat{P}_{21} & \hat{P}_{22} & \hat{P}_{23}\\ \hat{P}_{31} & \hat{P}_{32} & \hat{P}_{33}\end{bmatrix}. \tag{7}$$

Individual elements of the matrix (6) in explicit form can be represented

$$\hat{U}_{11}=(3-4\nu)+\frac{\vec{r}_1^{\,2}}{\vec{r}^{\,2}},\quad \hat{U}_{12}=\frac{\vec{r}_1\vec{r}_2}{\vec{r}^{\,2}},\quad \hat{U}_{13}=\frac{\vec{r}_1\vec{r}_3}{\vec{r}^{\,2}},\quad \hat{U}_{21}=\frac{\vec{r}_2\vec{r}_1}{\vec{r}^{\,2}},$$

$$\hat{U}_{22}=(3-4\nu)+\frac{\vec{r}_2^{\,2}}{\vec{r}^{\,2}},\quad \hat{U}_{23}=\frac{\vec{r}_2\vec{r}_3}{\vec{r}^{\,2}},\quad \hat{U}_{31}=\frac{\vec{r}_3\vec{r}_1}{\vec{r}^{\,2}},\quad \hat{U}_{32}=\frac{\vec{r}_3\vec{r}_2}{\vec{r}^{\,2}},$$

$$\hat{U}_{33}=(3-4\nu)+\frac{\vec{r}_3^{\,2}}{\vec{r}^{\,2}},$$

whilst from the matrix (7) take the following forms

$$\hat{P}_{11}=(B+3\frac{\vec{r}_1^{\,2}}{\vec{r}^{\,2}})\frac{\partial\vec{r}}{\partial n},\qquad \hat{P}_{22}=(B+3\frac{\vec{r}_2^{\,2}}{\vec{r}^{\,2}})\frac{\partial\vec{r}}{\partial n},$$

$$\hat{P}_{33}=(B+3\frac{\vec{r}_3^{\,2}}{\vec{r}^{\,2}})\frac{\partial\vec{r}}{\partial n},$$

$$\hat{P}_{12}=A\frac{\vec{r}_1\vec{r}_2}{\vec{r}^{\,2}}-B\frac{\vec{r}_1n_2-\vec{r}_2n_1}{\vec{r}},\quad \hat{P}_{13}=A\frac{\vec{r}_1\vec{r}_3}{\vec{r}^{\,2}}-B\frac{\vec{r}_1n_3-\vec{r}_3n_1}{\vec{r}}$$

$$\hat{P}_{21}=A\frac{\vec{r}_2\vec{r}_1}{\vec{r}^{\,2}}-B\frac{\vec{r}_2n_1-\vec{r}_1n_2}{\vec{r}}\quad \hat{P}_{23}=A\frac{\vec{r}_2\vec{r}_3}{\vec{r}^{\,2}}-B\frac{\vec{r}_2n_3-\vec{r}_3n_2}{\vec{r}},$$

$$\hat{P}_{31}=A\frac{\vec{r}_3\vec{r}_1}{\vec{r}^{\,2}}-B\frac{\vec{r}_3n_1-\vec{r}_1n_3}{\vec{r}},\quad \hat{P}_{32}=A\frac{\vec{r}_3\vec{r}_2}{\vec{r}^{\,2}}-B\frac{\vec{r}_3n_2-\vec{r}_2n_3}{\vec{r}},$$

where $A=3\frac{\partial\vec{r}}{\partial n}$, $B=(1-2\nu)$ and $\vec{r}_1=P_j^{(1)}(v,w)-x_1$, $\vec{r}_2=P_j^{(2)}(v,w)-x_2$, $\vec{r}_3=P_j^{(3)}(v,w)-x_3$, $\vec{r}=\left[\vec{r}_1^{\,2}+\vec{r}_2^{\,2}+\vec{r}_3^{\,2}\right]^{0.5}$, whilst $\boldsymbol{x}\equiv\{x_1,x_2,x_3\}$.

Appendix B Stresses in a domain

In order to obtain stress components we should use the following expression

$$\boldsymbol{\sigma}(\boldsymbol{x})=\sum_{j=1}^{n}\int_{v_{j-1}}^{v_j}\int_{w_{j-1}}^{w_j}\{\hat{\bar{\boldsymbol{D}}}_j^{*}(\boldsymbol{x},v,w)\,\boldsymbol{p}_j(v,w)-\hat{\bar{\boldsymbol{S}}}_j^{*}(\boldsymbol{x},v,w)\boldsymbol{u}_j(v,w)\}J_j(v,w)\,dv\,dw, \tag{8}$$

where integrands from (8) can be expressed

$$\hat{\bar{\boldsymbol{D}}}_j^{*}(\boldsymbol{x},v,w)=\frac{G}{8\pi(1-\nu)\vec{r}^{\,2}}\begin{bmatrix}\hat{D}_{111} & \hat{D}_{112} & \hat{D}_{113}\\ \hat{D}_{121} & \hat{D}_{122} & \hat{D}_{123}\\ \hat{D}_{131} & \hat{D}_{132} & \hat{D}_{133}\\ \hat{D}_{211} & \hat{D}_{212} & \hat{D}_{213}\\ \hat{D}_{221} & \hat{D}_{222} & \hat{D}_{223}\\ \hat{D}_{231} & \hat{D}_{232} & \hat{D}_{233}\\ \hat{D}_{311} & \hat{D}_{312} & \hat{D}_{313}\\ \hat{D}_{321} & \hat{D}_{322} & \hat{D}_{323}\\ \hat{D}_{331} & \hat{D}_{332} & \hat{D}_{333}\end{bmatrix} \tag{9}$$

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$$\hat{\bar{S}}_j^*(x,v,w) = \frac{G}{4\pi(1-\nu)\bar{r}^3}\begin{bmatrix} \hat{S}_{111} & \hat{S}_{112} & \hat{S}_{113} \\ \hat{S}_{121} & \hat{S}_{122} & \hat{S}_{123} \\ \hat{S}_{131} & \hat{S}_{132} & \hat{S}_{133} \\ \hat{S}_{211} & \hat{S}_{212} & \hat{S}_{213} \\ \hat{S}_{221} & \hat{S}_{222} & \hat{S}_{223} \\ \hat{S}_{231} & \hat{S}_{232} & \hat{S}_{233} \\ \hat{S}_{311} & \hat{S}_{312} & \hat{S}_{313} \\ \hat{S}_{321} & \hat{S}_{322} & \hat{S}_{323} \\ \hat{S}_{331} & \hat{S}_{332} & \hat{S}_{333} \end{bmatrix}. \quad (10)$$

Elements of the matrix (9) in explicit form can be presented

$\hat{D}_{111} = B\overleftrightarrow{r}_1 + 3\overleftrightarrow{r}_1^{\,3}$, $\hat{D}_{112} = -B\overleftrightarrow{r}_2 + 3\overleftrightarrow{r}_1^{\,2}\overleftrightarrow{r}_2$, $\hat{D}_{113} = -B\overleftrightarrow{r}_3 + 3\overleftrightarrow{r}_1^{\,2}\overleftrightarrow{r}_3$,

$\hat{D}_{121} = B\overleftrightarrow{r}_2 + 3\overleftrightarrow{r}_1^{\,2}\overleftrightarrow{r}_2$, $\hat{D}_{122} = B\overleftrightarrow{r}_1 + 3\overleftrightarrow{r}_1\overleftrightarrow{r}_2^{\,2}$, $\hat{D}_{123} = 3\overleftrightarrow{r}_1\overleftrightarrow{r}_2\overleftrightarrow{r}_3$,

$\hat{D}_{131} = B\overleftrightarrow{r}_3 + 3\overleftrightarrow{r}_1^{\,2}\overleftrightarrow{r}_3$, $\hat{D}_{132} = 3\overleftrightarrow{r}_1\overleftrightarrow{r}_3\overleftrightarrow{r}_2$, $\hat{D}_{133} = B\overleftrightarrow{r}_1 + 3\overleftrightarrow{r}_1\overleftrightarrow{r}_3^{\,2}$,

$\hat{D}_{211} = B\overleftrightarrow{r}_2 + 3\overleftrightarrow{r}_2\overleftrightarrow{r}_1^{\,2}$, $\hat{D}_{212} = B\overleftrightarrow{r}_1 + 3\overleftrightarrow{r}_2^{\,2}\overleftrightarrow{r}_1$, $\hat{D}_{213} = 3\overleftrightarrow{r}_2\overleftrightarrow{r}_1\overleftrightarrow{r}_3$,

$\hat{D}_{221} = -B\overleftrightarrow{r}_1 + 3\overleftrightarrow{r}_2^{\,2}\overleftrightarrow{r}_1$, $\hat{D}_{222} = B\overleftrightarrow{r}_2 + 3\overleftrightarrow{r}_2^{\,3}$, $\hat{D}_{223} = -B\overleftrightarrow{r}_3 + 3\overleftrightarrow{r}_2^{\,2}\overleftrightarrow{r}_3$,

$\hat{D}_{231} = 3\overleftrightarrow{r}_2\overleftrightarrow{r}_3\overleftrightarrow{r}_1$, $\hat{D}_{232} = B\overleftrightarrow{r}_3 + 3\overleftrightarrow{r}_2^{\,2}\overleftrightarrow{r}_3$, $\hat{D}_{233} = B\overleftrightarrow{r}_2 + 3\overleftrightarrow{r}_2\overleftrightarrow{r}_3^{\,2}$,

$\hat{D}_{311} = B\overleftrightarrow{r}_3 + 3\overleftrightarrow{r}_3\overleftrightarrow{r}_1^{\,2}$, $\hat{D}_{312} = 3\overleftrightarrow{r}_3\overleftrightarrow{r}_1\overleftrightarrow{r}_2$, $\hat{D}_{313} = B\overleftrightarrow{r}_1 + 3\overleftrightarrow{r}_3^{\,2}\overleftrightarrow{r}_1$,

$\hat{D}_{321} = 3\overleftrightarrow{r}_3\overleftrightarrow{r}_2\overleftrightarrow{r}_1$, $\hat{D}_{322} = B\overleftrightarrow{r}_3 + 3\overleftrightarrow{r}_3\overleftrightarrow{r}_2^{\,2}$, $\hat{D}_{323} = B\overleftrightarrow{r}_2 + 3\overleftrightarrow{r}_3^{\,2}\overleftrightarrow{r}_2$,

$\hat{D}_{331} = -B\overleftrightarrow{r}_1 + 3\overleftrightarrow{r}_3^{\,2}\overleftrightarrow{r}_1$, $\hat{D}_{332} = -B\overleftrightarrow{r}_2 + 3\overleftrightarrow{r}_3^{\,2}\overleftrightarrow{r}_2$, $\hat{D}_{333} = B\overleftrightarrow{r}_3 + 3\overleftrightarrow{r}_3^{\,3}$,

whilst in the matrix (10) can be presented by

$$\hat{S}_{111} = A(B\overleftrightarrow{r}_1 + 2\nu\overleftrightarrow{r}_1 - 5\overleftrightarrow{r}_1^{\,3}) + 6\nu n_1\overleftrightarrow{r}_1^{\,2} + B(3n_1\overleftrightarrow{r}_1^{\,2} + 2n_1) - Cn_1,$$

$$\hat{S}_{112} = A(B\overleftrightarrow{r}_2 - 5\overleftrightarrow{r}_1^{\,2}\overleftrightarrow{r}_2) + 6\nu n_1\overleftrightarrow{r}_1\overleftrightarrow{r}_2 + B(3n_2\overleftrightarrow{r}_1^{\,2}) - Cn_2,$$

$$\hat{S}_{113} = A(B\overleftrightarrow{r}_3 - 5\overleftrightarrow{r}_1^{\,2}\overleftrightarrow{r}_3) + 6\nu n_1\overleftrightarrow{r}_1\overleftrightarrow{r}_3 + B(3n_3\overleftrightarrow{r}_1^{\,2}) - Cn_3,$$

$$\hat{S}_{121} = A(\nu\overleftrightarrow{r}_2 - 5\overleftrightarrow{r}_1^{\,2}\overleftrightarrow{r}_2) + 3\nu(n_1\overleftrightarrow{r}_2\overleftrightarrow{r}_1 + n_2\overleftrightarrow{r}_1^{\,2}) + B(3n_1\overleftrightarrow{r}_1\overleftrightarrow{r}_2 + n_2),$$

$$\hat{S}_{122} = A(\nu\overleftrightarrow{r}_1 - 5\overleftrightarrow{r}_1\overleftrightarrow{r}_2^{\,2}) + 3\nu(n_1\overleftrightarrow{r}_2^{\,2} + n_2\overleftrightarrow{r}_1\overleftrightarrow{r}_2) + B(3n_2\overleftrightarrow{r}_1\overleftrightarrow{r}_2 + n_1),$$

$$\hat{S}_{123} = A(-5\overleftrightarrow{r}_1\overleftrightarrow{r}_2\overleftrightarrow{r}_3) + 3\nu(n_1\overleftrightarrow{r}_2\overleftrightarrow{r}_3 + n_2\overleftrightarrow{r}_1\overleftrightarrow{r}_3) + B(3n_3\overleftrightarrow{r}_1\overleftrightarrow{r}_2),$$

$$\hat{S}_{131} = A(\nu\overleftrightarrow{r}_3 - 5\overleftrightarrow{r}_1^{\,2}\overleftrightarrow{r}_3) + 3\nu(n_1\overleftrightarrow{r}_3\overleftrightarrow{r}_1 + n_3\overleftrightarrow{r}_1^{\,2}) + B(3n_1\overleftrightarrow{r}_1\overleftrightarrow{r}_3 + n_3),$$

$$\hat{S}_{132} = A(-5\overleftrightarrow{r}_1\overleftrightarrow{r}_2\overleftrightarrow{r}_3) + 3\nu(n_1\overleftrightarrow{r}_3\overleftrightarrow{r}_2 + n_3\overleftrightarrow{r}_1\overleftrightarrow{r}_2) + B(3n_2\overleftrightarrow{r}_1\overleftrightarrow{r}_3),$$

$$\hat{S}_{133} = A(\nu\overleftrightarrow{r}_1 - 5\overleftrightarrow{r}_1\overleftrightarrow{r}_3^{\,2}) + 3\nu(n_1\overleftrightarrow{r}_3^{\,2} + n_3\overleftrightarrow{r}_1\overleftrightarrow{r}_3) + B(3n_3\overleftrightarrow{r}_1\overleftrightarrow{r}_3 + n_1),$$

$$\hat{S}_{211} = A(\nu\overleftrightarrow{r}_2 - 5\overleftrightarrow{r}_2\overleftrightarrow{r}_1^{\,2}) + 3\nu(n_2\overleftrightarrow{r}_1^{\,2} + n_1\overleftrightarrow{r}_2\overleftrightarrow{r}_1) + B(3n_1\overleftrightarrow{r}_2\overleftrightarrow{r}_1 + n_2),$$

$$\hat{S}_{212} = A(\nu\overleftrightarrow{r}_1 - 5\overleftrightarrow{r}_2^{\,2}\overleftrightarrow{r}_1) + 3\nu(n_2\overleftrightarrow{r}_1\overleftrightarrow{r}_2 + n_1\overleftrightarrow{r}_2^{\,2}) + B(3n_2\overleftrightarrow{r}_2\overleftrightarrow{r}_1 + n_1),$$

$$\hat{S}_{213} = A(-5\overleftrightarrow{r}_2\overleftrightarrow{r}_1\overleftrightarrow{r}_3) + 3\nu(n_2\overleftrightarrow{r}_1\overleftrightarrow{r}_3 + n_1\overleftrightarrow{r}_2\overleftrightarrow{r}_3) + B(3n_3\overleftrightarrow{r}_2\overleftrightarrow{r}_1),$$

$$\hat{S}_{221} = A(B\overleftrightarrow{r}_1 - 5\overleftrightarrow{r}_2^{\,2}\overleftrightarrow{r}_1) + 6\nu n_2\overleftrightarrow{r}_2\overleftrightarrow{r}_1 + B(3n_1\overleftrightarrow{r}_1\overleftrightarrow{r}_2 + 2n_1) - Cn_1,$$

$$\hat{S}_{222} = A(B\overleftrightarrow{r}_2 + 2\nu\overleftrightarrow{r}_2 - 5\overleftrightarrow{r}_2^{\,3}) + 6\nu n_2\overleftrightarrow{r}_2^{\,2} + B(3n_2\overleftrightarrow{r}_2^{\,2} + 2n_2) - Cn_2,$$

$$\hat{S}_{223} = A(B\overleftrightarrow{r}_3 - 5\overleftrightarrow{r}_2^{\,2}\overleftrightarrow{r}_3) + 6\nu n_2\overleftrightarrow{r}_2\overleftrightarrow{r}_3 + B(3n_3\overleftrightarrow{r}_2^{\,2}) - Cn_3,$$

$$\hat{S}_{231} = A(-5\overleftrightarrow{r}_2\overleftrightarrow{r}_3\overleftrightarrow{r}_1) + 3\nu(n_2\overleftrightarrow{r}_3\overleftrightarrow{r}_1 + n_3\overleftrightarrow{r}_2\overleftrightarrow{r}_1) + B(3n_1\overleftrightarrow{r}_2\overleftrightarrow{r}_3),$$

$$\hat{S}_{232} = A(\nu\overleftrightarrow{r}_3 - 5\overleftrightarrow{r}_2^{\,2}\overleftrightarrow{r}_3) + 3\nu(n_2\overleftrightarrow{r}_3\overleftrightarrow{r}_2 + n_3\overleftrightarrow{r}_2^{\,2}) + B(3n_2\overleftrightarrow{r}_2\overleftrightarrow{r}_3 + n_3),$$

$$\hat{S}_{233} = A(\nu\overleftrightarrow{r}_2 - 5\overleftrightarrow{r}_2\overleftrightarrow{r}_3^{\,2}) + 3\nu(n_2\overleftrightarrow{r}_3^{\,2} + n_3\overleftrightarrow{r}_2\overleftrightarrow{r}_3) + B(3n_3\overleftrightarrow{r}_2\overleftrightarrow{r}_3 + n_2),$$

$$\hat{S}_{311} = A(\nu\overleftrightarrow{r}_3 - 5\overleftrightarrow{r}_3\overleftrightarrow{r}_1^{\,2}) + 3\nu(n_3\overleftrightarrow{r}_1^{\,2} + n_1\overleftrightarrow{r}_3\overleftrightarrow{r}_1) + B(3n_1\overleftrightarrow{r}_3\overleftrightarrow{r}_1 + n_3),$$

$$\hat{S}_{312} = A(-5\overleftrightarrow{r}_3\overleftrightarrow{r}_1\overleftrightarrow{r}_2) + 3\nu(n_3\overleftrightarrow{r}_1\overleftrightarrow{r}_2 + n_1\overleftrightarrow{r}_3\overleftrightarrow{r}_2) + B(3n_2\overleftrightarrow{r}_3\overleftrightarrow{r}_1),$$

$$\hat{S}_{313} = A(\nu\overleftrightarrow{r}_1 - 5\overleftrightarrow{r}_3^{\,2}\overleftrightarrow{r}_1) + 3\nu(n_3\overleftrightarrow{r}_1\overleftrightarrow{r}_3 + n_1\overleftrightarrow{r}_3^{\,2}) + B(3n_3\overleftrightarrow{r}_3\overleftrightarrow{r}_1 + n_1),$$

$$\hat{S}_{321} = A(-5\overleftrightarrow{r}_3\overleftrightarrow{r}_2\overleftrightarrow{r}_1) + 3\nu(n_3\overleftrightarrow{r}_2\overleftrightarrow{r}_1 + n_2\overleftrightarrow{r}_3\overleftrightarrow{r}_1) + B(3n_1\overleftrightarrow{r}_3\overleftrightarrow{r}_2),$$

$$\hat{S}_{322} = A(\nu\overleftrightarrow{r}_3 - 5\overleftrightarrow{r}_3\overleftrightarrow{r}_2^{\,2}) + 3\nu(n_3\overleftrightarrow{r}_2^{\,2} + n_2\overleftrightarrow{r}_3\overleftrightarrow{r}_2) + B(3n_2\overleftrightarrow{r}_3\overleftrightarrow{r}_2 + n_3),$$

$$\hat{S}_{323} = A(\nu\overleftrightarrow{r}_2 - 5\overleftrightarrow{r}_3^{\,2}\overleftrightarrow{r}_2) + 3\nu(n_3\overleftrightarrow{r}_2\overleftrightarrow{r}_3 + n_2\overleftrightarrow{r}_3^{\,2}) + B(3n_3\overleftrightarrow{r}_3\overleftrightarrow{r}_2 + n_2),$$

$$\hat{S}_{331} = A(B\overleftrightarrow{r}_1 - 5\overleftrightarrow{r}_3^{\,2}\overleftrightarrow{r}_1) + 6\nu n_3\overleftrightarrow{r}_3\overleftrightarrow{r}_1 + B(3n_1\overleftrightarrow{r}_3^{\,2}) - Cn_1,$$

$$\hat{S}_{332} = A(B\overleftrightarrow{r}_2 - 5\overleftrightarrow{r}_3^{\,2}\overleftrightarrow{r}_2) + 6\nu n_3\overleftrightarrow{r}_3\overleftrightarrow{r}_2 + B(3n_2\overleftrightarrow{r}_3^{\,2}) - Cn_2,$$

$$\hat{S}_{333} = A(B\overleftrightarrow{r}_3 + 2\nu\overleftrightarrow{r}_3 - 5\overleftrightarrow{r}_3^{\,3}) + 6\nu n_3\overleftrightarrow{r}_3^{\,2} + B(3n_3\overleftrightarrow{r}_3^{\,2} + 2n_3) - Cn_3,$$

where $C = (1-4\nu)$.

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